Journal of Algebra 321 (2009) 2861–2874

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Simplicity of vacuum modules over affine Lie superalgebras ☆

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ARTICLE INFO

Article history:
Received 16 June 2008
Available online 20 February 2009
Communicated by Vera Serganova

Keywords:
Affine Lie superalgebra
Vacuum module
W-algebra

ABSTRACT

We prove an explicit condition on the level k for the irreducibility of a vacuum module V^k over a (non-twisted) affine Lie superalgebra, which was conjectured by M. Gorelik and V.G. Kac. An immediate consequence of this work is the simplicity conditions for the corresponding minimal W-algebras obtained via quantum reduction, in all cases except when the level k is a non-negative integer.

Introduction

The (non-twisted) affine Lie superalgebra $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K + \mathbb{C}D$ is obtained from a simple finite dimensional Lie superalgebra $\mathfrak{g}$, with a non-degenerate even invariant bilinear form $B(\cdot,\cdot)$, and has the following commutation relations:

$$\left[at^m, bt^n\right] = [a,b]t^{m+n} + m\delta_{m,-n}B(a,b)K, \qquad \left[D, at^m\right] = -mat^m, \qquad [K, \hat{\mathfrak{g}}] = 0.$$

Let $2h_B^\vee$ be the eigenvalue of the Casimir operator $\sum_i a_i a^i$ in the adjoint representation, where $\{a_i\}$ and $\{a^i\}$ are dual bases of $\mathfrak{g}$ with respect to $B(\cdot,\cdot)$.

The vacuum module over $\hat{\mathfrak{g}}$ is the induced module

$$V^k = \operatorname{Ind}_{\mathfrak{g}[t]+\mathbb{C}K+\mathbb{C}D}^{\hat{\mathfrak{g}}} \mathbb{C}_k,$$

☆ Work supported in part by Minerva grant, No. 8596/1.
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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.02.003

where $\mathbb{C}_k$ is the 1-dimensional module with trivial action of $\mathfrak{g}[t] + \mathbb{C}D$, and the action of K is given by $k \cdot \mathrm{Id}$ for some $k \in \mathbb{C}$. Note that this module does not depend on the choice of simple roots of $\mathfrak{g}$.

We prove the following theorem, which was conjectured by M. Gorelik and V.G. Kac [7].

Theorem 0.1. *Let $\mathfrak{g}$ be an (almost) simple finite dimensional Lie superalgebra of positive defect. Then the $\hat{\mathfrak{g}}$-module V^k is not irreducible if and only if*

$$\frac{k + h_B^\vee}{B(\alpha,\alpha)} \in \mathbb{Q}_{\geqslant 0}$$

for some even root α of $\mathfrak{g}$.

It was shown in [7] that Theorem 0.1 holds for simple Lie superalgebras with defect zero or one, using the Shapovalov determinant. In this paper, we prove the theorem for simple Lie superalgebras with defect greater than or equal to two, completing the proof of the theorem.

Fix $(\cdot,\cdot)$ to be the non-degenerate even invariant bilinear form on $\mathfrak{g}$ with standard normalization as introduced in [11]. Then $h^\vee$ is called the dual Coxeter number of $\mathfrak{g}$ (see [11] for the values). Then $(\alpha,\alpha) \in \mathbb{Q}$ for $\alpha \in \Delta$. When the defect is greater than or equal to two, there exist even roots α and α' such that $(\alpha,\alpha) > 0$ and $(\alpha',\alpha') < 0$. In this case, Theorem 0.1 can be reformulated as follows. Let $\mathfrak{g}$ be an (almost) simple Lie superalgebra with defect greater than or equal to two. A vacuum module V^k of $\hat{\mathfrak{g}}$ is irreducible if and only if $k \in \mathbb{C} \setminus \mathbb{Q}$.

Our proof goes as follows. Let V^k be a vacuum module over $\hat{\mathfrak{g}}$. By analyzing a character formula given in [7], we show that if the level k is a rational number then the Jantzen filtration is non-trivial. For each superalgebra, our proof is broken up into two cases, namely $k + h^\vee > 0$ and $k + h^\vee < 0$. For each case, we choose a different set of simple roots for the finite dimensional Lie superalgebra $\mathfrak{g}$. Note that a vacuum module is always reducible at the critical level $k = -h^\vee$. The fact that V^k is simple for $k \notin \mathbb{Q}$ follows from the vacuum determinant, and is shown in [7].

An application of the main theorem is given in the last section. We obtain simplicity conditions for the minimal W-algebras $W^k(\mathfrak{g}, f_\theta)$, where $\mathfrak{g}$ is a simple contragredient finite dimensional Lie superalgebra, f_θ is a root vector of the lowest root, which is assumed to be even, and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geqslant 0}$. This is achieved via quantum reduction, which was introduced for Lie algebras in [3,4] and extended to Lie superalgebras in [10].

1. Preliminaries

1.1. Affine Lie superalgebras

Let $\mathfrak{g}$ be a simple finite dimensional Lie superalgebra, and let $\hat{\mathfrak{g}}$ be the corresponding affine Lie superalgebra [8,9]. Let Δ (resp. $\hat{\Delta}$) denote the roots of $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$). We use the standard notations for roots. Corresponding to a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of $\mathfrak{g}$, we have the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Let $\hat{\Pi} = \{\alpha_0 := \delta - \theta\} \cup \Pi$ be the simple roots of $\hat{\mathfrak{g}}$, where θ is the highest root of $\mathfrak{g}$, and let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+$ be the corresponding triangular decomposition, where $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$. The root lattice of $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$) is defined to be $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$ (resp. $\hat{Q} = \sum_{i=0}^n \mathbb{Z}\alpha_i$). Let $Q^+ = \sum_{i=1}^n \mathbb{N}\alpha_i$ and $\hat{Q}^+ = \sum_{i=0}^n \mathbb{N}\alpha_i$. Define a partial ordering on $\mathfrak{h}^*$ by $\alpha \geqslant \beta$ if $\alpha - \beta \in Q^+$.

Let $\kappa(\cdot,\cdot)$ denote the Killing form of $\mathfrak{g}$. If κ is non-zero, set $\Delta^\# = \{\alpha \in \Delta \mid \kappa(\alpha,\alpha) > 0\}$. If $\kappa = 0$, then $\mathfrak{g}$ is of type $A(n|n)$, $D(n+1|n)$ or $D(1,2,\alpha)$. In this case, Δ_0 is a union of two orthogonal subsystems: $\Delta_0 = A_n \cup A_n$, $D_{n+1} \cup C_n$, $D_2 \cup C_1$, respectively, and we let $\Delta^\#$ be the first subset. Let $W^\#$ be the subgroup of the Weyl group W generated by the reflections r_α with $\alpha \in \Delta^\#$. Then $W^\#$ is the Weyl group for the root system $\Delta^\#$.

Let $(\cdot,\cdot)$ denote the non-degenerate symmetric even invariant bilinear form on $\mathfrak{g}$, which is normalized by the condition $(\alpha,\alpha) = 2$ for a long root α of $\Delta^\#$. Since $\mathfrak{g}$ is simple, the Killing form $\kappa(\cdot,\cdot)$ is proportional the standard form $(\cdot,\cdot)$. However, it is possible that $\kappa = 0$. We can extend this form to $\hat{\mathfrak{g}}$ as follows:

$$\big(at^n, bt^m\big) = \delta_{n,-m}(a,b), \quad a, b \in \mathfrak{g};$$

$$\big(\mathbb{C}K + \mathbb{C}D, \mathfrak{g}\big[t, t^{-1}\big]\big) = 0; \qquad (K,K) = (D,D) = 0; \qquad (K,D) = 1.$$

Choose $\rho \in \mathfrak{h}^*$ (resp. $\hat{\rho} \in \hat{\mathfrak{h}}^*$) such that $(\rho, \alpha_j) = \frac{1}{2}(\alpha_j, \alpha_j)$ for $\alpha_j \in \Pi$ (resp. $\alpha_j \in \hat{\Pi}$). Note that for $\nu \in \mathfrak{h}^*$ we have that $(\hat{\rho}, \nu) = (\rho, \nu)$. Recall [8] that

$$(\delta, \hat{\rho}) = h^\vee = (\rho, \theta) + \frac{1}{2}(\theta, \theta). \tag{1}$$

Define $\Lambda_0 \in \hat{\mathfrak{h}}^*$ by $\Lambda_0(h) = 0$ for $h \in \mathfrak{h} \oplus \mathbb{C}D$ and $\Lambda_0(K) = 1$.

The Weyl denominator of $\mathfrak{g}$ is

$$R := \prod_{\alpha \in \Delta_{\bar{0}}^+} \big(1 - e^{-\alpha}\big) \prod_{\alpha \in \Delta_{\bar{1}}^+} \big(1 + e^{-\alpha}\big)^{-1} = \sum_{\eta \in Q^+} k_\Pi(\eta) e^{-\eta}, \quad \text{where } k_\Pi(\eta) \in \mathbb{Z}.$$

The function k_Π is extended to $\hat{Q}$, by setting $k_\Pi(\eta) = 0$ for $\eta \in \hat{Q} \setminus Q^+$ [11].

1.2. Vacuum modules

The vacuum module $V^k := \mathrm{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g} + \hat{\mathfrak{n}}^+ + \hat{\mathfrak{h}}} \mathbb{C}_k$ is a generalized Verma module $M_I(\lambda)$ with $\lambda = k\Lambda_0$ and $I \subseteq J := \{0, 1, \ldots, n\}$ corresponding to Π (see [7]). The $\hat{\mathfrak{g}}$-module $M_I(\lambda)$ is the quotient of the Verma module $M(\lambda)$ by the submodule $\mathcal{U}(\hat{\mathfrak{g}})\mathfrak{n}^- v_\lambda$, where v_λ is the highest weight vector of $M(\lambda)$. So V^k has a unique maximal submodule.

Let

$$\mathrm{Irr} := \left\{ \alpha \in \hat{Q}^+ \setminus Q \;\middle|\; \frac{\alpha}{n} \notin \hat{Q}^+ \text{ for } n \in \mathbb{Z}_{\geqslant 2} \right\}$$

and

$$C(\lambda) := \left\{ (m, \xi) \in \mathbb{Z}_{\geqslant 1} \times \mathrm{Irr} \;\middle|\; (\lambda + \hat{\rho}, m\xi) - \frac{1}{2}(m\xi, m\xi) = 0 \right\}. \tag{2}$$

Let $\mathcal{F}^i(V^k)$, $i \in \mathbb{N}$ be the Jantzen filtration of the module V^k. Then by [7]

$$\sum_{i=1}^{\infty} \mathrm{ch}\, \mathcal{F}^i\big(M_I(\lambda)\big) = \sum_{(m,\xi) \in C(\lambda)} a_{m,\xi}\, \mathrm{ch}\, M(\lambda - m\xi) \tag{3}$$

where

$$a_{m,\xi} = \sum_{\gamma \in \hat{\Delta}^+ \setminus \Delta} \sum_{r=1}^{\infty} (-1)^{(r+1)p(\gamma)} (\dim \mathfrak{g}_\gamma) k_\Pi(m\xi - r\gamma). \tag{4}$$

Lemma 1.1. *The module V^k is not simple if and only if there exists $(m, \xi) \in C(\lambda)$ such that $a_{m,\xi} \neq 0$.*

Proof. Recall that $\mathcal{F}^1(V^k)$ is the maximal submodule of V^k. Hence, the module V^k is simple if and only if its Jantzen filtration is trivial. Since the characters of different Verma modules are linearly independent, it follows from (3) that the filtration is non-trivial if and only if there exists $(m, \xi) \in C(\lambda)$ such that $a_{m,\xi} \neq 0$. $\square$

Lemma 1.2. *Let $\mu \in Q^+$. If $k_\Pi(\mu) \neq 0$, then $(\rho, \mu) = \frac{1}{2}(\mu, \mu)$.*

Proof. One has that

$$\begin{aligned} \operatorname{ch} V^k &= \operatorname{ch} M(k\Lambda_0) \cdot \frac{\prod_{\Delta_{\bar{0}}^+}(1-e^{-\alpha})}{\prod_{\Delta_{\bar{1}}^+}(1+e^{-\alpha})} \\ &= \operatorname{ch} M(k\Lambda_0) \cdot \sum_{\mu \in Q^+} k_\Pi(\mu) e^{-\mu} \\ &= \sum_{\mu \in Q^+} k_\Pi(\mu) \operatorname{ch} M(k\Lambda_0 - \mu). \end{aligned}$$

The character of a highest weight module can be uniquely written as a linear combination of characters of Verma modules, and the Casimir operator gives the same scalar on each of these Verma modules [8]. Hence, if $k_\Pi(\mu) \neq 0$ then

$$(k\Lambda_0 + \hat{\rho}, k\Lambda_0 + \hat{\rho}) = (k\Lambda_0 - \mu + \hat{\rho}, k\Lambda_0 - \mu + \hat{\rho}),$$

which implies that $(\rho, \mu) = \frac{1}{2}(\mu, \mu)$. $\square$

1.3. The Weyl denominator expansion

The aim of this section is to expand R using the Weyl denominator identity given in [11]. For a finite set $X := \{\lambda_i\}_{i=1}^r \subset \hat{\mathfrak{h}}^*$, let C_X be the collection of elements of the form $\sum_{i=1}^r \sum_{\mu < \lambda_i} c_\mu e^\mu$, where $c_\mu \in \mathbb{Z}$. Let C be the union of all C_X over finite subsets of $\hat{\mathfrak{h}}^*$ [6]. Note that $x, y \in C$ implies $x + y, xy \in C$. We will expand R to an element of C.

The *defect* of $\mathfrak{g}$, denoted by $\operatorname{def}\mathfrak{g}$, is the dimension of a maximal isotropic subspace of $\mathfrak{h}_{\mathbb{R}}^* := \sum_{\alpha \in \Delta} \mathbb{R}\alpha$. A subset S of Δ is called *isotropic* if it spans an isotropic subspace of $\mathfrak{h}_{\mathbb{R}}^*$. It is called *maximal isotropic* if $|S| = \operatorname{def}\mathfrak{g}$. By [11], one can always choose a set of simple roots that contains a given maximal isotropic set S. Fix a set of simple roots Π which contains a maximal isotropic set S. Denote $\mathbb{Z}S := \{\sum_{\beta \in S} n_\beta \beta \mid n_\beta \in \mathbb{Z}\}$ and $\mathbb{N}S := \{\sum_{\beta \in S} n_\beta \beta \mid n_\beta \in \mathbb{N}\}$. For $\mu = \sum_{\beta \in S} n_\beta \beta \in \mathbb{N}S$, define the height of μ to be $\operatorname{ht}\mu = \sum n_\beta$.

For $w \in W^\#$, let

$$T_w = \{\beta \in S \mid w(\beta) \in \Delta^-\},$$

and define $|w| \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}S, Q)$ such that for $\beta \in S \subset \Pi$,

$$|w|(\beta) := \begin{cases} -w(\beta), & \text{if } \beta \in T_w; \\ w(\beta), & \text{if } \beta \notin T_w. \end{cases}$$

Note that $|w|(\mu) \in Q^+$ for any $\mu \in \mathbb{N}S$. Define $\varphi : W^\# \to -Q^+$ by

$$\varphi(w) := \sum_{\beta \in T_w} w(\beta).$$

Lemma 1.3. *Suppose Π contains a maximal isotropic set S. Let R be the Weyl denominator of $\mathfrak{g}$. Then*

$$R = \sum_{\eta \in Q^+} k_\Pi(\eta) e^{-\eta} = \sum_{w \in W^\#} \sum_{\mu \in \mathbb{N}S} (-1)^{l(w) + \operatorname{ht}\mu} e^{\varphi(w) - |w|(\mu) + w(\rho) - \rho}.$$

Proof. The assertion follows from the following computations:

$$\begin{aligned}
R &\overset{\text{by [11]}}{=} \sum_{w\in W^\#} (-1)^{l(w)} w\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right)e^{-\rho} \\
&= \sum_{w\in W^\#} (-1)^{l(w)} \frac{e^{w(\rho)-\rho}}{\prod_{\beta\in S}(1+e^{-w(\beta)})} \\
&= \sum_{w\in W^\#} (-1)^{l(w)} \frac{e^{w(\rho)-\rho+\varphi(w)}}{\prod_{\beta\in S}(1+e^{-|w|(\beta)})} \\
&= \sum_{w\in W^\#} \sum_{\mu\in\mathbb{N}S} (-1)^{l(w)+\operatorname{ht}\mu} e^{\varphi(w)-|w|(\mu)+w(\rho)-\rho}. \qquad \square
\end{aligned}$$

Corollary 1.4. *Suppose Π contains a maximal isotropic set S. If $k_\Pi(\eta)\neq 0$, then there exists $w\in W^\#$ and $\mu\in\mathbb{N}S$ such that*

$$-\eta = \varphi(w) - |w|(\mu) + w(\rho) - \rho.$$

2. Root systems

In this section, we describe the root systems of the simple finite dimensional Lie superalgebras which appear in the present paper [5]. A root system of a simple finite dimensional Lie superalgebra $\mathfrak{g}$ is described in terms of a basis $\{\varepsilon_i, \delta_j \mid 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n\}$, with the bilinear form $(\cdot,\cdot)$ normalized such that $(\alpha,\alpha)=2$ for a long root $\alpha\in\Delta^\#$. We can identify $\mathfrak{h}^*$ with a linear subspace of $V:=\operatorname{span}\{\varepsilon_1,\dots,\varepsilon_m,\delta_1,\dots,\delta_n\}$, and write $\mu\in\mathfrak{h}^*$ as $\mu=\sum_{i=1}^m c_{\varepsilon_i}(\mu)\varepsilon_i+\sum_{j=1}^n c_{\delta_j}(\mu)\delta_j$, with coefficients $c_{\varepsilon_i}(\mu), c_{\delta_j}(\mu)\in\mathbb{C}$.

For $A(m-1|n-1)=\mathfrak{sl}(m|n)$, we identify $\mathfrak{h}^*$ with the linear subspace of V given by

$$\mathfrak{h}^* = \left\{a_1\varepsilon_1+\cdots+a_m\varepsilon_m+b_1\delta_1+\cdots+b_n\delta_n \;\middle|\; \sum_{i=1}^m a_i+\sum_{j=1}^n b_j=0\right\},$$

and choose the normalization

$$(\varepsilon_i,\varepsilon_j)=\delta_{ij}, \qquad (\delta_i,\delta_j)=-\delta_{ij}, \qquad (\varepsilon_i,\delta_j)=0.$$

The root system is $\Delta=\Delta_{\bar 0}\cup\Delta_{\bar 1}$ where

$$\begin{aligned}
\Delta_{\bar 0} &= \{\varepsilon_i-\varepsilon_j \mid 1\leqslant i<j\leqslant m\}\cup\{\delta_i-\delta_j \mid 1\leqslant i<j\leqslant n\}, \\
\Delta_{\bar 1} &= \left\{\pm(\varepsilon_i-\delta_j) \;\middle|\; 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n\right\}.
\end{aligned}$$

We may assume without loss of generality that $m\geqslant n$, since $A(m-1|n-1)\cong A(n-1|m-1)$. Thus,

$$\Delta^\# = \{\varepsilon_i-\varepsilon_j \mid 1\leqslant i,j\leqslant m,\ i\neq j\}.$$

We extend the action of $W^\#$ to $\operatorname{span}\{\varepsilon_1,\dots,\varepsilon_m\}$ by the trivial action on $\sum_{i=1}^m\varepsilon_i$. Then $W^\#$ is the permutation group of $\{\varepsilon_1,\dots,\varepsilon_m\}$.

For $B(m|n)=\mathfrak{osp}(2m+1|2n)$ with $m\geqslant n+1$, we identify $\mathfrak{h}^*$ with V, and choose the normalization

$$(\varepsilon_i,\varepsilon_j)=\delta_{ij}, \qquad (\delta_i,\delta_j)=-\delta_{ij}, \qquad (\varepsilon_i,\delta_j)=0.$$

The root system is $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ where

$$\Delta_{\bar{0}} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm\delta_k \pm \delta_l, \pm 2\delta_k \mid 1 \leqslant i < j \leqslant m,\ 1 \leqslant k < l \leqslant n\},$$
$$\Delta_{\bar{1}} = \{\pm\varepsilon_i \pm \delta_j, \pm\delta_j \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}.$$

Thus,

$$\Delta^{\#} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid 1 \leqslant i < j \leqslant m\}.$$

Then $W^{\#}$ is the group of signed permutations of $\{\varepsilon_1, \dots, \varepsilon_m\}$.

For $B(n|m) = \mathfrak{osp}(2n+1|2m)$ with $m \geqslant n$, we identify $\mathfrak{h}^*$ with V, and choose the normalization

$$(\varepsilon_i, \varepsilon_j) = \frac{1}{2}\delta_{ij}, \qquad (\delta_i, \delta_j) = -\frac{1}{2}\delta_{ij}, \qquad (\varepsilon_i, \delta_j) = 0.$$

The root system is $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ where

$$\Delta_{\bar{0}} = \left\{\pm\delta'_i \pm \delta'_j, \pm\delta'_i, \pm\varepsilon'_k \pm \varepsilon'_l, \pm 2\varepsilon'_k \;\middle|\; 1 \leqslant i < j \leqslant m,\ 1 \leqslant k < l \leqslant n\right\},$$
$$\Delta_{\bar{1}} = \left\{\pm\delta'_i \pm \varepsilon'_j, \pm\varepsilon'_j \;\middle|\; 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\right\}.$$

Here

$$\Delta^{\#} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \leqslant i < j \leqslant m\}.$$

Then $W^{\#}$ is the group of signed permutations of $\{\varepsilon_1, \dots, \varepsilon_m\}$.

For $D(m|n) = \mathfrak{osp}(2m|2n)$ with $m \geqslant n+1$, we identify $\mathfrak{h}^*$ with V, and choose the normalization

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \qquad (\delta_i, \delta_j) = -\delta_{ij}, \qquad (\varepsilon_i, \delta_j) = 0.$$

The root system is $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ where

$$\Delta_{\bar{0}} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\delta_k \pm \delta_l, \pm 2\delta_k \mid 1 \leqslant i < j \leqslant m,\ 1 \leqslant k < l \leqslant n\},$$
$$\Delta_{\bar{1}} = \{\pm\varepsilon_i \pm \delta_k \mid 1 \leqslant i \leqslant m,\ 1 \leqslant k \leqslant n\}.$$

Thus,

$$\Delta^{\#} = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant m\}.$$

Then $W^{\#}$ is the group of signed permutations of $\{\varepsilon_1, \dots, \varepsilon_m\}$ which change an even number of the signs.

For $D(n|m) = \mathfrak{osp}(2n|2m)$ with $m \geqslant n$, we identify $\mathfrak{h}^*$ with V, and choose the normalization

$$(\varepsilon_i, \varepsilon_j) = \frac{1}{2}\delta_{ij}, \qquad (\delta_i, \delta_j) = -\frac{1}{2}\delta_{ij}, \qquad (\varepsilon_i, \delta_j) = 0.$$

The root system is $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ where

$$\Delta_{\bar{0}} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i, \pm\delta_k \pm \delta_l \mid 1 \leqslant i < j \leqslant m,\ 1 \leqslant k < l \leqslant n\},$$
$$\Delta_{\bar{1}} = \{\pm\varepsilon_i \pm \delta_k \mid 1 \leqslant i \leqslant m,\ 1 \leqslant k \leqslant n\}.$$

Here

$$\Delta^{\#} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \leqslant i < j \leqslant m\}.$$

Then $W^{\#}$ is the group of signed permutations of $\{\varepsilon_1, \ldots, \varepsilon_m\}$.

The defect is n for all of the Lie superalgebras described above, so we assume that $n \geqslant 2$. We extend the action of $W^{\#}$ to V by the trivial action on the linear span of $\{\delta_1, \ldots, \delta_n\}$.

3. Simplicity of vacuum modules

3.1. Preliminaries

Let $\mathfrak{g}$ be an (almost) simple finite dimensional Lie superalgebra with bilinear form $(\cdot,\cdot)$ normalized by the condition that $(\alpha, \alpha) = 2$ for a long root of $\Delta^{\#}$. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$. Note that $h^\vee \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ (see Table II). Fix a set of simple roots $\Pi = \{\beta_1, \ldots, \beta_n\}$ and denote the highest weight θ.

Lemma 3.1. *Suppose that (θ, θ) is a non-zero integer. Let $k \in \mathbb{Q}$ such that $\frac{k+h^\vee}{(\theta,\theta)} > 0$. Choose $q \in 2\mathbb{Z}_{\geqslant 1}$ such that $q(\frac{k+h^\vee}{(\theta,\theta)}) \in \mathbb{Z}$ and $q > \frac{(\rho,\theta)}{k+h^\vee}$. Define*

$$N := 2q\left(\frac{k + h^\vee}{(\theta, \theta)}\right) - \frac{2(\rho, \theta)}{(\theta, \theta)}. \tag{5}$$

1. *If $\frac{\theta}{2} \notin \Delta$ and $\frac{2(\rho,\theta)}{(\theta,\theta)} \in \mathbb{Z}$, then $(N, q\delta - \theta) \in C(k\Lambda_0)$.*
2. *If $\frac{\theta}{2} \in \Delta$ and $\frac{2(\rho,\theta)}{(\theta,\theta)} \in \frac{1}{2} + \mathbb{Z}$, then $(2N, \frac{q}{2}\delta - \frac{\theta}{2}) \in C(k\Lambda_0)$ and $2N$ is an odd integer.*

Proof. By Table II, $N \in \mathbb{Z}_{\geqslant 1}$ in the first case, while $2N \in \mathbb{Z}_{\geqslant 1}$ in the second case. Also, $q\delta - \theta \in \hat{Q}^+ \setminus Q$ and

$$\big(k\Lambda_0 + \hat{\rho}, N(q\delta - \theta)\big) - \frac{1}{2}\big(N(q\delta - \theta), N(q\delta - \theta)\big) = N\left(qk + qh^\vee - (\rho, \theta) - \frac{N}{2}(\theta, \theta)\right) = 0.$$

Hence, the lemma follows from (2). $\square$

Express $\theta = \sum_{i=1}^n b_i\beta_i$ with $b_i \in \mathbb{Z}_{\geqslant 1}$, and let $b' = \max\{b_1, \ldots, b_n\}$.

Lemma 3.2. *Suppose that $N, q, r, l \in \mathbb{Z}_{\geqslant 1}$, $\alpha \in \{0\} \cup \Delta \setminus \{\theta\}$ and $(N(q\delta - \theta) - r(l\delta - \alpha)) \in Q^+$. Then*

1. *$Nq = rl$, $r\alpha - N\theta \in Q^+$, and $\alpha \in \Delta^+$;*
2. *$\sum_{i=1}^n \beta_i \leqslant \alpha$;*
3. *$r - N \geqslant \frac{1}{b'}N > 0$.*

Proof. Statement 1 follows immediately. For 2, express $\alpha = \sum_{i=1}^n a_i\beta_i$ with $a_i \in \mathbb{Z}_{\geqslant 0}$. Now $r\alpha - N\theta \in Q^+$ implies that $0 \leqslant ra_i - Nb_i$ for $i = 1, \ldots, n$. Hence, $a_i \geqslant 1$ for $i = 1, \ldots, n$. For 3, since $0 < \alpha < \theta$, we have that $a_i \leqslant b_i$ for $i = 1, \ldots, n$ and there is an index j such that $a_j \leqslant b_j - 1$. Thus,

$$N \leqslant ra_j - N(b_j - 1) \leqslant (r - N)a_j \leqslant (r - N)b'. \qquad \square$$

3.2. Proof of the main theorem

Theorem 3.3. *Let $\mathfrak{g}$ be an (almost) simple finite dimensional Lie superalgebra with defect greater than or equal to two. If $k \in \mathbb{Q}$, then the vacuum module V^k over $\hat{\mathfrak{g}}$ is not simple.*

Proof. Let $\mathfrak{g}$ be an (almost) simple finite dimensional Lie superalgebra with defect greater than or equal to two. Fix $k \in \mathbb{Q}$. Now $k \in \mathbb{Q}$ if and only if $k + h^\vee \in \mathbb{Q}$, since $h^\vee \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$. If $k = -h^\vee$ then V^k is not simple, so we assume now that $k + h^\vee \in \mathbb{Q} \setminus \{0\}$. If $\mathfrak{g} = D(n+2|n)$ then we assume that $\frac{1}{k+h^\vee} \notin \mathbb{Z}_{\geqslant 1}$. We will handle this case separately. Let Π be the set of simple roots listed in Table I corresponding to $\mathfrak{g}$ and $k + h^\vee$. We have chosen Π so that the highest weight θ satisfies the conditions: $(\theta, \theta) \neq 0$ and $\frac{k+h^\vee}{(\theta,\theta)} > 0$, (see Table II).

By Lemma 1.1, it suffices to show that there exists $(m, \xi) \in C(k\Lambda_0)$ such that $a_{m,\xi} \neq 0$ in (3). By (4), it suffices to find $(m, \xi) \in C(k\Lambda_0)$ with $\xi \in \hat{\Delta}^+$ such that for all $(r, \gamma) \in \mathbb{Z}_{\geqslant 1} \times (\hat{\Delta}^+ \setminus \Delta)$, we have that (m, ξ) satisfies the conditions:

1. if $(r, \gamma) \neq (m, \xi)$, then $r\gamma \neq m\xi$,
2. if $r\gamma \neq m\xi$, then $k_\Pi(m\xi - r\gamma) = 0$.

Indeed, in this case

$$a_{m,\xi} = (-1)^{(m+1)p(\xi)} \dim \mathfrak{g}_\xi,$$

which is non-zero.

Choose $q \in 2\mathbb{Z}_{\geqslant 1}$ such that $q(\frac{k+h^\vee}{(\theta,\theta)}) \in \mathbb{Z}_{\geqslant 1}$ and $q > \frac{(\rho,\theta)}{k+h^\vee}$. Define N as in (5). Note that for each $n \in \mathbb{Z}$ it is possible to choose q sufficiently large such that $N > n$. So we may assume that $N \gg 0$.

By Table II, if $\frac{\theta}{2} \notin \Delta$ then $\frac{2(\rho,\theta)}{(\theta,\theta)} \in \mathbb{Z}$. Then by Lemma 3.1, $(N, q\delta - \theta) \in C(k\Lambda_0)$. Since $c\theta \notin \Delta^+$ for $c \neq 1$, we have that $(N, q\delta - \theta)$ satisfies condition 1. If $\frac{\theta}{2} \in \Delta$, then $\frac{2(\rho,\theta)}{(\theta,\theta)} \in \frac{1}{2} + \mathbb{Z}$ (see Table II). Then by Lemma 3.1, $(2N, \frac{q}{2}\delta - \frac{1}{2}\theta) \in C(k\Lambda_0)$ and $2N$ is an odd integer. Since $c\frac{\theta}{2} \notin \Delta^+$ for $c \notin \{1, 2\}$ and $2N$ is odd, we have that $(2N, \frac{q}{2}\delta - \frac{1}{2}\theta)$ satisfies condition 1.

Suppose that $k_\Pi(N(q\delta - \theta) - r\gamma) \neq 0$ for some $(r, \gamma) \in \mathbb{Z}_{\geqslant 1} \times (\hat{\Delta}^+ \setminus \Delta)$ such that $(r, \gamma) \neq (N, q\delta - \theta)$. Write $\gamma = l\delta - \alpha$ for some $l \in \mathbb{Z}_{\geqslant 1}$ and $\alpha \in \Delta \cup \{0\}$.

Case 1. Suppose $\alpha \neq \theta, \frac{\theta}{2}$. By Lemma 3.2, we have $\sum_{\alpha_i \in \Pi} \alpha_i \leqslant \alpha$ and $r - N \geqslant \frac{1}{b'}N > 0$. Hence, we may assume that $r - N \gg 0$. Also, $Nq = rl$, which implies $k_\Pi(r\alpha - N\theta) \neq 0$. Thus by Lemma 1.2,

$$2(\rho, r\alpha - N\theta) = (r\alpha - N\theta, r\alpha - N\theta),$$

implying

$$(\alpha, \alpha)r^2 + \big(-2(\rho, \alpha) - 2N(\alpha, \theta)\big)r + N^2(\theta, \theta) + 2N(\rho, \theta) = 0. \tag{6}$$

Subcase 1. If $(\alpha, \alpha) \neq 0$, the discriminant D for this quadratic equation in the variable r is

$$\begin{aligned} D &= \big(2(\rho, \alpha) + 2N(\alpha, \theta)\big)^2 - 2(\alpha, \alpha)\big(N^2(\theta, \theta) + 2N(\rho, \theta)\big) \\ &= 4N^2\big((\alpha, \theta)(\alpha, \theta) - (\alpha, \alpha)(\theta, \theta)\big) + 8N\big((\rho, \alpha)(\alpha, \theta) - (\alpha, \alpha)(\rho, \theta)\big) + 4(\rho, \alpha)^2. \end{aligned}$$

By Lemma 4.2,

$$(\alpha, \alpha)(\theta, \theta) > (\alpha, \theta)(\alpha, \theta),$$

which implies that $D < 0$ for $N \gg 0$. This contradicts the assumption that r is an integer.

Subcase 2. If $(\alpha,\alpha)=0$, then by solving (6) we obtain

$$r=\frac{N^2(\theta,\theta)+2N(\rho,\theta)}{2N(\theta,\alpha)+2(\rho,\alpha)}. \tag{7}$$

Note that the denominator is non-zero for N sufficiently large. Indeed, by Lemma 4.1, $2(\theta,\alpha)=(\theta,\theta)$. By substituting this into (7), we obtain

$$r=\frac{N^2(\theta,\theta)+2N(\rho,\theta)}{N(\theta,\theta)+2(\rho,\alpha)}=N+\frac{2((\rho,\theta)-(\rho,\alpha))}{(\theta,\theta)+(\frac{2(\rho,\alpha)}{N})}.$$

Since $r>N$ we have that $(\rho,\theta)\neq(\rho,\alpha)$. If $(\rho,\alpha)\neq 0$, then $r\notin\mathbb{Z}$ for $N\gg 0$. If $(\rho,\alpha)=0$, then

$$r=N+\frac{2(\rho,\theta)}{(\theta,\theta)}.$$

But by Lemma 3.2, $r-N>\frac{2(\rho,\theta)}{(\theta,\theta)}$ for $N\gg 0$, which is a contradiction.

Case 2. Suppose $\alpha=c\theta$. Then $k_\Pi((rc-N)\theta)\neq 0$ and $rc>N$. By Lemma 1.2,

$$2\big(\rho,(rc-N)\theta\big)=\big((rc-N)\theta,(rc-N)\theta\big),$$

which implies that

$$rc-N=\frac{2(\rho,\theta)}{(\theta,\theta)}. \tag{8}$$

Hence, $\frac{2(\rho,\theta)}{(\theta,\theta)}>0$. Then $(\theta,\theta)=2$ by Table II. Now $(\rho,\theta)\neq 0$ and $k_\Pi(\frac{2(\rho,\theta)}{(\theta,\theta)}\theta)\neq 0$, so it follows from Lemma 4.3 that $\mathfrak{g}=D(n+2|n)$.

If $\mathfrak{g}=D(n+2|n)$, then $\theta=\varepsilon_1+\varepsilon_2$ and $\alpha=c\theta\in\Delta^+$ implies that $c=1$. Then by Table II and (8) we have $r=N+1$. Now $N(q\delta-\theta)-r(l\delta-\theta)\in Q^+$ implies that $Nq=rl$. After substituting $r=N+1$ we have

$$Nq=(N+1)l. \tag{9}$$

Since N and $N+1$ are relatively prime, $N+1$ divides q. Hence, there exists $d\in\mathbb{Z}_{\geqslant 1}$ such that

$$q=(N+1)d. \tag{10}$$

By substituting the values given in Table II into (5), we have

$$N=q\big(k+h^\vee\big)-1. \tag{11}$$

Combining (10) and (11) we obtain

$$d=\frac{1}{k+h^\vee},$$

where $d\in\mathbb{Z}_{\geqslant 1}$. But we assumed that if $\mathfrak{g}=D(n+2|n)$, then $\frac{1}{k+h^\vee}\notin\mathbb{Z}_{\geqslant 1}$.

Case $\mathfrak{g} = D(n+2|n)$ and $d := \frac{1}{k+h^\vee} \in \mathbb{Z}_{\geqslant 1}$. Choose a maximal isotropic set

$$S = \{\varepsilon_i - \delta_i \mid 1 \leqslant i \leqslant n\}$$

and a set of simple roots

$$\{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \delta_n - \varepsilon_{n+1}, \varepsilon_{n+1} - \varepsilon_{n+2}, \varepsilon_{n+1} + \varepsilon_{n+2}\}$$

which contains S. Then $\theta = \varepsilon_1 + \delta_1$, $(\theta, \theta) = 0$ and $(\theta, \rho) = 2$.

We will show that $a_{N, q\delta - \theta} \neq 0$, where $q := 2d$ and $N \in \mathbb{Z}_{\geqslant 1}$ with $N \gg 0$. It will then follow from Lemma 1.1 that V^k is not simple. First, note that $(N, q\delta - \theta) \in C(k\Lambda_0)$ for any N, since by the definition of q we have

$$\big(k\Lambda_0 + \hat{\rho}, N(q\delta - \theta)\big) - \frac{1}{2}\big(N(q\delta - \theta), N(q\delta - \theta)\big) = N\big(q\big(k + h^\vee\big) - (\rho, \theta)\big) = 0.$$

Now $q\delta - \theta \in \hat{\Delta}^+ \setminus \Delta$. If $(r, \gamma) \in \mathbb{Z}_{\geqslant 1} \times (\hat{\Delta}^+ \setminus \Delta)$ such that $r\gamma = N(q\delta - \theta)$, then $(r, \gamma) = (N, q\delta - \theta)$.

Suppose that $k_\Pi(N(q\delta - \theta) - r\gamma) \neq 0$ for some $(r, \gamma) \in \mathbb{Z}_{\geqslant 1} \times (\hat{\Delta}^+ \setminus \Delta)$ such that $(r, \gamma) \neq (N, q\delta - \theta)$. Write $\gamma = l\delta - \alpha$ for some $l \in \mathbb{Z}_{\geqslant 1}$ and $\alpha \in \Delta \cup \{0\}$.

If $\alpha = \theta$, then $k_\Pi((r - N)\theta) \neq 0$ and $r > N$. By Lemma 1.2,

$$\big(\rho, (r - N)\theta\big) = \big((r - N)\theta, (r - N)\theta\big).$$

But $(\theta, \theta) = 0$ and $(\rho, \theta) = 2$, so this is a contradiction.

Now assume that $\alpha \neq \theta$. Then by Lemma 3.2, $Nq = rl$, $r > N$, $k_\Pi(r\alpha - N\theta) \neq 0$, and

$$\alpha \in \left\{\beta \in \Delta \,\middle|\, \sum_{\alpha_i \in \Pi} \alpha_i \leqslant \beta < \theta\right\} = \{\varepsilon_1 + \varepsilon_i, \varepsilon_1 + \delta_j \mid 2 \leqslant i \leqslant n+1,\ 2 \leqslant j \leqslant n\},$$

and for all $\alpha \in A_\Pi$ we have that $(\alpha, \theta) = 1$, $(\alpha, \rho) = 2$ and $(\alpha, \alpha) \in \{0, 2\}$.

By Lemma 1.2,

$$(\rho, r\alpha - N\theta) = \frac{1}{2}(r\alpha - N\theta, r\alpha - N\theta)$$

implying

$$r^2\frac{(\alpha, \alpha)}{2} - rN(\alpha, \theta) - r(\alpha, \rho) + N(\theta, \rho) + N^2\frac{(\theta, \theta)}{2} = 0.$$

After substituting we have

$$\frac{(\alpha, \alpha)}{2}r^2 - (N + 2)r + 2N = 0.$$

If $(\alpha, \alpha) = 0$ then $r = \frac{2N}{N+2} \notin \mathbb{Z}$ for $N \gg 0$, which is a contradiction. If $(\alpha, \alpha) = 2$, then $r \in \{2, N\}$. But we have that $r > N$ and $N \gg 0$, so this is also a contradiction. $\square$

Table I

$\mathfrak{g}$		$k+h^\vee$	Π
$A(m-1\|n-1)$,	$m \geqslant n$	$+$	$\{\varepsilon_1-\delta_1, \delta_1-\delta_2, \delta_2-\varepsilon_2, \varepsilon_2-\delta_3, \delta_3-\varepsilon_3, \ldots, \delta_n-\varepsilon_n, \varepsilon_n-\varepsilon_{n+1}, \ldots, \varepsilon_{m-1}-\varepsilon_m\}$
$A(m-1\|n-1)$,	$m \geqslant n$	$-$	$\{\delta_1-\varepsilon_1, \varepsilon_1-\varepsilon_2, \ldots, \varepsilon_{m-n+1}-\varepsilon_{m-n+2}, \varepsilon_{m-n+2}-\delta_2, \delta_2-\varepsilon_{m-n+3},$ $\varepsilon_{m-n+3}-\delta_3, \ldots, \varepsilon_m-\delta_n\}$
$B(m\|n)$,	$m=n+1$	$+$	$\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \ldots, \varepsilon_{n+1}-\delta_n, \delta_n\}$
$B(m\|n)$,	$m=n+2$	$+$	$\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \ldots, \varepsilon_{n+1}-\delta_n, \delta_n-\varepsilon_{n+2}, \varepsilon_{n+2}\}$
$B(m\|n)$,	$m \geqslant n+3$	$+$	$\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \ldots, \varepsilon_{n+1}-\delta_n, \delta_n-\varepsilon_{n+2}, \varepsilon_{n+2}-\varepsilon_{n+3}, \ldots, \varepsilon_{m-1}-\varepsilon_m, \varepsilon_m\}$
$B(m\|n)$,	$m \geqslant n+1$	$-$	$\{\delta_1-\delta_2, \ldots, \delta_{n-1}-\delta_n, \delta_n-\varepsilon_1, \varepsilon_1-\varepsilon_2, \ldots, \varepsilon_{m-1}-\varepsilon_m, \varepsilon_m\}$
$B(n\|m)$,	$m=n$	$+$	$\{\varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \ldots, \varepsilon_n-\delta_n, \delta_n\}$
$B(n\|m)$,	$m=n+1$	$+$	$\{\varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \ldots, \varepsilon_n-\delta_n, \delta_n-\varepsilon_{n+1}, \varepsilon_{n+1}\}$
$B(n\|m)$,	$m \geqslant n+2$	$+$	$\{\varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \ldots, \varepsilon_n-\delta_n, \delta_n-\varepsilon_{n+1}, \varepsilon_{n+1}-\varepsilon_{n+2}, \ldots, \varepsilon_{m-1}-\varepsilon_m, \varepsilon_m\}$
$B(n\|m)$,	$m \geqslant n$	$-$	$\{\delta_1-\delta_2, \delta_2-\delta_3, \ldots, \delta_n-\varepsilon_1, \varepsilon_1-\varepsilon_2, \ldots, \varepsilon_{m-1}-\varepsilon_m, \varepsilon_m\}$
$D(m\|n)$,	$m=n+1$	$+$	$\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\delta_2, \delta_2-\varepsilon_3, \varepsilon_3-\delta_3, \delta_3-\varepsilon_4, \ldots, \varepsilon_n-\delta_n, \delta_n-\varepsilon_{n+1}, \delta_n+\varepsilon_{n+1}\}$
$D(m\|n)$,	$m=n+2$	$+$ $\frac{1}{k+h^\vee} \notin \mathbb{Z}$	$\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\varepsilon_3, \ldots, \delta_n-\varepsilon_{n+2}, \delta_n+\varepsilon_{n+2}\}$
$D(m\|n)$,	$m \geqslant n+3$	$+$	$\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\varepsilon_3, \ldots, \delta_n-\varepsilon_{n+2}, \varepsilon_{n+2}-\varepsilon_{n+3}, \ldots, \varepsilon_{m-1}-\varepsilon_m, \varepsilon_{m-1}+\varepsilon_m\}$
$D(m\|n)$,	$m \geqslant n+1$	$-$	$\{\delta_1-\varepsilon_1, \varepsilon_1-\delta_2, \delta_2-\varepsilon_2, \varepsilon_2-\delta_3, \ldots, \delta_n-\varepsilon_n, \varepsilon_n-\varepsilon_{n+1}, \ldots, \varepsilon_{m-1}-\varepsilon_m, \varepsilon_{m-1}+\varepsilon_m\}$
$D(n\|m)$,	$m=n$	$+$	$\{\varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \varepsilon_2-\delta_2, \ldots, \varepsilon_n-\delta_n, \varepsilon_n+\delta_n\}$
$D(n\|m)$,	$m \geqslant n+1$	$+$	$\{\varepsilon_1-\delta_1, \delta_1-\varepsilon_2, \varepsilon_2-\delta_2, \ldots, \varepsilon_n-\delta_n, \delta_n-\varepsilon_{n+1}, \varepsilon_{n+1}-\varepsilon_{n+2}, \ldots, \varepsilon_{m-1}-\varepsilon_m, 2\varepsilon_m\}$
$D(n\|m)$,	$m \geqslant n$	$-$	$\{\delta_1-\delta_2, \delta_2-\delta_3, \ldots, \delta_n-\varepsilon_1, \varepsilon_1-\varepsilon_2, \ldots, \varepsilon_{m-1}-\varepsilon_m, 2\varepsilon_m\}$

Table II

$\mathfrak{g}$		$k+h^\vee$	$h^\vee$	θ	(θ,θ)	$\frac{2(\rho,\theta)}{(\theta,\theta)}$	$\frac{\theta}{2}$
$A(m-1\|n-1)$,	$m \geqslant n$	$+$	$m-n$	$\varepsilon_1-\varepsilon_m$	2	$m-n-1$	
$A(m-1\|n-1)$,	$m \geqslant n$	$-$	$m-n$	$\delta_1-\delta_n$	-2	$-m+n-1$	
$B(m\|n)$,	$m \geqslant n+1$	$+$	$2(m-n)-1$	$\varepsilon_1+\varepsilon_2$	2	$2m-2n-2$	
$B(m\|n)$,	$m \geqslant n+1$	$-$	$2(m-n)-1$	$2\delta_1$	-4	$-m+n-\frac{1}{2}$	δ_1
$B(n\|m)$,	$m \geqslant n$	$+$	$m-n+\frac{1}{2}$	$2\varepsilon_1$	2	$m-n-\frac{1}{2}$	ε_1
$B(n\|m)$,	$m \geqslant n$	$-$	$m-n+\frac{1}{2}$	$\delta_1+\delta_2$	-1	$-2m+2n-2$	
$D(m\|n)$,	$m \geqslant n+1$	$+$	$2(m-n-1)$	$\varepsilon_1+\varepsilon_2$	2	$2m-2n-3$	
$D(m\|n)$,	$m \geqslant n+1$	$-$	$2(m-n-1)$	$2\delta_1$	-4	$-m+n$	
$D(n\|m)$,	$m \geqslant n$	$+$	$m-n+1$	$2\varepsilon_1$	2	$m-n$	
$D(n\|m)$,	$m \geqslant n$	$-$	$m-n+1$	$\delta_1+\delta_2$	-1	$-2m+2n-3$	

Table III

$\mathfrak{g}$		$k+h^\vee$	θ	A_Π
$A(m-1\|n-1)$,	$m \geqslant n$	$+$	$\varepsilon_1-\varepsilon_m$	$\emptyset$
$A(m-1\|n-1)$,	$m \geqslant n$	$-$	$\delta_1-\delta_m$	$\emptyset$
$B(m\|n)$,	$m \geqslant n+1$	$+$	$\varepsilon_1+\varepsilon_2$	$\{\varepsilon_1, \varepsilon_1+\varepsilon_i, \varepsilon_1+\delta_j\}_{i=3,\ldots,m,\ j=1,\ldots,n}$
$B(m\|n)$,	$m \geqslant n+1$	$-$	$2\delta_1$	$\{\delta_1, \delta_1+\varepsilon_i, \delta_1+\delta_j\}_{i=1,\ldots,m-1,\ j=2,\ldots,n}$
$B(n\|m)$,	$m \geqslant n$	$+$	$2\varepsilon_1$	$\{\varepsilon_1, \varepsilon_1+\varepsilon_i, \varepsilon_1+\delta_j\}_{i=2,\ldots,m,\ j=1,\ldots,n}$
$B(n\|m)$,	$m \geqslant n$	$-$	$\delta_1+\delta_2$	$\{\delta_1+\varepsilon_i, \delta_1+\delta_j\}_{i=1,\ldots,m-1,\ j=3,\ldots,n}$
$D(m\|n)$,	$m \geqslant n+1$	$+$	$\varepsilon_1+\varepsilon_2$	$\{(\varepsilon_1+\varepsilon_i), (\varepsilon_1+\delta_j)\}_{i=3,\ldots,m-1,\ j=1,\ldots,n}$
$D(m\|n)$,	$m \geqslant n+1$	$-$	$2\delta_1$	$\{(\delta_1+\varepsilon_i), (\delta_1+\delta_j)\}_{i=1,\ldots,m-1,\ j=2,\ldots,n}$
$D(n\|m)$,	$m \geqslant n$	$+$	$2\varepsilon_1$	$\{(\varepsilon_1+\varepsilon_i), (\varepsilon_1+\delta_j)\}_{i=2,\ldots,m,\ j=1,\ldots,n}$
$D(n\|m)$,	$m \geqslant n$	$-$	$\delta_1+\delta_2$	$\{(\delta_1+\varepsilon_i), (\delta_1+\delta_j)\}_{i=1,\ldots,m,\ j=3,\ldots,n}$

4. Tables and computations

Table I records our choice of simple roots Π for each of our cases. The defect is n, so we assume $n \geqslant 2$. When $k+h^\vee > 0$ we write "+", and when $k+h^\vee < 0$ we write "−".

Table II records properties of Π. We indicate when $\frac{\theta}{2}$ is a root.

In Table III let $A_\Pi = \{\alpha \in \Delta \mid \sum_{\alpha_i \in \Pi} \alpha_i \leqslant \alpha < \theta\}$.

Table IV

$\mathfrak{g}$		θ	S
$A(m-1\|n-1)$,	$m \geqslant n$	$\varepsilon_1 - \varepsilon_m$	$\beta_1 = \varepsilon_1 - \delta_1$, $\beta_i = \delta_i - \varepsilon_i$ for $i = 2, \ldots, n$
$B(m\|n)$, $D(m\|n)$,	$m \geqslant n+2$	$\varepsilon_1 + \varepsilon_2$	$\beta_i = \varepsilon_{i+1} - \delta_i$ for $i = 1, \ldots, n$
$B(n\|m)$, $D(n\|m)$,	$m \geqslant n+1$	$2\varepsilon_1$	$\beta_i = \varepsilon_i - \delta_i$ for $i = 1, \ldots, n$

Lemma 4.1. *Let Π be one of the sets of simple roots in Table* I. *If $\alpha \in A_\Pi$, then*

$$2(\alpha, \theta) = (\theta, \theta).$$

Proof. This calculation follows from Table III. $\square$

Lemma 4.2. *Let Π be one of the sets of simple roots in Table* I. *If $\alpha \in A_\Pi$ such that $\alpha \neq \frac{\theta}{2}$ and $(\alpha, \alpha) \neq 0$, then*

$$(\alpha, \alpha)(\theta, \theta) > (\alpha, \theta)(\alpha, \theta).$$

Proof. By our choice of simple roots, $(\theta, \theta) \neq 0$. From Table III we see that (α, α) has the same sign as (θ, θ). If $\alpha \in A_\Pi$ such that $\alpha \neq \frac{\theta}{2}$ and $(\alpha, \alpha) \neq 0$, then $|(\alpha, \alpha)| \in \{1, 2, 4\}$. If $4(\alpha, \alpha)(\theta, \theta) \leqslant (\theta, \theta)(\theta, \theta)$, then $|(\alpha, \alpha)| = 1$ and $|(\theta, \theta)| = 4$. By Table III, this implies that $\alpha = \delta_1$ and $\theta = 2\delta_1$. But this contradicts the assumption that $\alpha \neq \frac{\theta}{2}$. Hence, the result follows from Lemma 4.1. $\square$

In Table IV we have chosen Π to contain a maximal isotropic subset $S = \{\beta_1, \ldots, \beta_n\}$ whenever (θ, θ) and (ρ, θ) are both positive.

Lemma 4.3. *Let Π be one of the sets of simple roots in Table* I, *excluding $D(n+2|n)$. If $(\rho, \theta) \neq 0$, then*

$$k_\Pi\left(\frac{2(\rho, \theta)}{(\theta, \theta)}\theta\right) = 0.$$

Proof. This is clear when $\frac{2(\rho,\theta)}{(\theta,\theta)} < 0$. Suppose $\frac{2(\rho,\theta)}{(\theta,\theta)} > 0$. Then $(\theta, \theta) = 2$ by Table II, which implies $(\rho, \theta) > 0$. We have chosen Π to contain a maximal isotropic subset S when (θ, θ) and (ρ, θ) are both positive (see Table IV).

Suppose that $k_\Pi((\rho, \theta)\theta) \neq 0$. Then by Corollary 1.4, there exists $w \in W^\#$ and $\mu \in \mathbb{N}S$ such that

$$-(\rho, \theta)\theta = \varphi(w) - |w|(\mu) + w(\rho) - \rho. \tag{12}$$

Write $\mu \in \mathbb{N}S$ as $\mu = \sum_{\beta \in S} b_\beta \beta$ where $b_\beta \in \mathbb{N}$. Then by definition

$$|w|(\mu) = \sum_{\beta \in S \setminus T_w} b_\beta w(\beta) - \sum_{\beta \in T_w} b_\beta w(\beta),$$

which implies

$$\varphi(w) - |w|(\mu) = \sum_{\beta \in T_w} (1 + b_\beta) w(\beta) - \sum_{\beta \in S \setminus T_w} b_\beta w(\beta). \tag{13}$$

Since coefficients $c_{\delta_j}(\theta)$ equal zero for $1 \leqslant j \leqslant n$, it follows from (12) that

$$c_{\delta_j}\big(\varphi(w) - |w|(\mu) + w(\rho) - \rho\big) = 0, \quad \text{for } 1 \leqslant j \leqslant n.$$

Since $w \in W^{\#}$ fixes $\delta_1, \dots, \delta_n$, the coefficients $c_{\delta_j}(w(\rho) - \rho)$ equal zero for $1 \leqslant j \leqslant n$. Thus,

$$c_{\delta_j}\big(\varphi(w) - |w|(\mu)\big) = 0, \quad \text{for } 1 \leqslant j \leqslant n. \tag{14}$$

Now $c_{\delta_j}(\beta_i) = 0$ when $j \neq i$, while $c_{\delta_i}(\beta_i) \neq 0$ (see Table IV). Since $w \in W^{\#}$ fixes $\delta_1, \dots, \delta_n$, we have that $c_{\delta_j}(w(\beta_i)) = 0$ when $j \neq i$, while $c_{\delta_i}(w(\beta_i)) \neq 0$. Then it follows from (14) that the coefficients in (13) must all be equal to zero. Since $b_\beta \geqslant 0$, this implies that $T_w = \emptyset$, $\mu = 0$, and $\varphi(w) - |w|(\mu) = 0$. Therefore,

$$w(\rho) - \rho = -(\rho, \theta)\theta \tag{15}$$

for some $w \in W^{\#}$ satisfying $T_w = \emptyset$.

Case 1. If $\beta_1 = \varepsilon_1 - \delta_1$, then $c_{\varepsilon_1}(\theta) \neq 0$ (see Table IV). Now $w(\beta_1) \in \Delta^+$ since $T_w = \emptyset$, which implies $w(\varepsilon_1) = \varepsilon_1$ (see Table I). Thus, $c_{\varepsilon_1}(w(\rho) - \rho) = 0$. Then (15) and $c_{\varepsilon_1}(\theta) \neq 0$ together imply that $(\rho, \theta) = 0$, which contradicts $(\rho, \theta) > 0$.

Case 2. If $\beta_1 = \varepsilon_2 - \delta_1$, then $\theta = \varepsilon_1 + \varepsilon_2$ and $\mathfrak{g}$ is either $B(m|n)$ or $D(m|n)$ with $m \geqslant n+2$ (see Table IV). Since T_w is empty we have $w(\varepsilon_2) \in \{\varepsilon_1, \varepsilon_2\}$. If $w(\varepsilon_2) = \varepsilon_2$, then $c_{\varepsilon_2}(w(\rho) - \rho) = 0$ and (15) does not hold since $\theta = \varepsilon_1 + \varepsilon_2$. If $w(\varepsilon_2) = \varepsilon_1$, then

$$c_{\varepsilon_1}\big(w(\rho) - \rho\big) = \big(\rho, w^{-1}(\varepsilon_1)\big) - (\rho, \varepsilon_1) = -(\rho, \varepsilon_1 - \varepsilon_2) = -1,$$

since $\varepsilon_1 - \varepsilon_2 \in \Pi$. Then (15) implies $(\rho, \theta) = 1$. Then by (1) it follows that $h^\vee = 2$. Then by Table II we see that $\mathfrak{g} = D(n+2|n)$. $\square$

5. Simplicity of minimal *W*-algebras

Let $\mathfrak{g}$ be a simple finite dimensional Lie superalgebra equipped with a non-degenerate even invariant bilinear form $B(\cdot,\cdot)$. Normalize $B(\cdot,\cdot)$ such that $B(\theta, \theta) = 2$ for the highest root θ, which is assumed to be even. Let f_θ be the lowest root vector of $\mathfrak{g}$. For each $k \in \mathbb{C}$, one can define a vertex algebra $W^k(\mathfrak{g}, f_\theta)$, called the minimal W-algebra, which is described in [10,12]. This class of W-algebras contains the well-known superconformal algebras, including the Virasoro algebra, the Bershadsky–Polyakov algebra, the Neveu–Schwarz algebra, the Bershadsky–Knizhnik algebras, and the $N = 2, 3, 4$ superconformal algebras. From the present work, we obtain a criterion for the simplicity of $W^k(\mathfrak{g}, f_\theta)$ when $k \notin \mathbb{Z}_{\geqslant 0}$.

Let $\hat{\mathfrak{g}}$ be the (non-twisted) affinization of $\mathfrak{g}$, and let $\mathcal{O}_k$ be the Bernstein–Gel'fand–Gel'fand category of $\hat{\mathfrak{g}}$ at level $k \in \mathbb{C}$ (see [2]). In [10,12], a functor from the category $\mathcal{O}_k$ to the category of $\mathbb{Z}$-graded $W^k(\mathfrak{g}, f_\theta)$-modules is given. This functor, which is referred to as quantum reduction, has many remarkable properties. In particular, it is proven in [1] that this functor H is exact and that $H(L(\lambda))$ is either irreducible or zero, where $L(\lambda)$ denotes the unique simple quotient of the Verma module $M(\lambda)$. The image of the vacuum module V^k under this functor is the vertex algebra $W^k(\mathfrak{g}, f_\theta)$, viewed as a module over itself.

Theorem 5.1. *(See M. Gorelik and V.G. Kac [7].)*

(i) *The vertex algebra $W^k(\mathfrak{g}, f_\theta)$ is simple if and only if the $\hat{\mathfrak{g}}$-module V^k is irreducible, or $k \in \mathbb{Z}_{\geqslant 0}$ and V^k has length two (i.e. the maximal proper submodule of the V^k is irreducible).*
(ii) *If $\mathfrak{g}$ is a simple Lie algebra, $\mathfrak{g} = \mathfrak{sl}_2$, then $W^k(\mathfrak{g}, f_\theta)$ is simple if and only if V^k is irreducible. This holds if and only if $\frac{k+h^\vee}{B(\alpha,\alpha)} \notin \mathbb{Q}_{\geqslant 0} \backslash \{\frac{1}{2m}\}_{m \in \mathbb{Z}_{\geqslant 1}}$ for a long root α.*

From Theorems 0.1 and 5.1, we deduce the following:

Corollary 5.2. *Let $\mathfrak{g}$ be a simple contragredient finite dimensional Lie superalgebra of positive defect and let $k \in \mathbb{C} \setminus \mathbb{Z}_{\geqslant 0}$. Then $W^k(\mathfrak{g}, f_\theta)$ is not simple if and only if*

$$\frac{k + h^\vee}{B(\alpha, \alpha)} \in \mathbb{Q}_{\geqslant 0}$$

for some even root α of $\mathfrak{g}$.

For affine Lie superalgebras, V^k is always reducible when $k \in \mathbb{Z}_{\geqslant 0}$. Thus, in order to determine the simplicity conditions for all minimal W-algebras, one is left with answering the following question.

Problem 5.3. Let $\mathfrak{g}$ be an affine Lie superalgebra and $k \in \mathbb{Z}_{\geqslant 0}$. Is the maximal submodule of V^k simple?

Acknowledgments

We would like to thank Professor Maria Gorelik and Professor Anthony Joseph for reading drafts of the paper and for helpful discussions.

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